I. INTRODUCTION

Equations that appear in certain fields of physics, or science in general, are often identical, and therefore, a given phenomenon tends to have analogous phenomena in different fields. The experiment you will conduct is the study of oscillating electric RCL circuits, which simulate mechanical oscillations.

II. THEORY

Consider the following circuit (Fig. 1):

![Electric RCL circuit diagram](image)

Recall the voltage-current relationships of the different electric elements:

For the resistance $R$ : 

$$U_R = I_R R$$

For the capacitor $C$ :

$$\int I_c dt = CU_C$$

For the inductance $L$ :

$$U_C = -L \frac{dI}{dt}$$

($U_x$ and $I_x$ represent the voltage difference and the current across an $x$ element)

Using the Kirchhoff laws, and defining $U_{out} = U(t) = U$ yields:

$$C \cdot U(t) = \int I_c \cdot dt \quad \text{et} \quad U(t) = R \cdot I_R$$

Since:

$$I_C + I_R = I_L$$

Then:

$$C \frac{dU}{dt} + \frac{U}{R} = \int \frac{U - U_{in}}{L} dt$$
Differentiating with respect to time yields:

\[ \ddot{U} + \frac{\dot{U}}{RC} + \frac{U}{LC} = \frac{U_{in}}{LC} \quad (1) \]

Equation (1) is similar to the equation defining a damped harmonic oscillator

**Model harmonic oscillator.**

Let’s consider the following model of a harmonic oscillator (fig. 2): a point mass m moves along an axis Ox (single degree of liberty). It is subject to:

1) A **spring force** \( \tilde{F}_s = -k \cdot \dot{x} \) that tries to bring the mass back to its equilibrium position.
2) A **viscous friction** :
   \( \tilde{F}_f = -\eta \cdot \dot{x} \)
3) An external time dependant **perturbation**

![Fig. 2: Damped oscillator](image)

The mass’ differential equation of motion is given by (to be shown by reader):

\[ \ddot{x} + 2\lambda \cdot \dot{x} + \omega_0^2 \cdot x = p \cdot \sin(\Omega \cdot t) \quad (2) \]

where: \( \lambda = \frac{\eta}{2m} \) \quad \( \omega_0^2 = \frac{k}{m} \) \quad \( \frac{G(t)}{m} = p \sin \Omega \cdot t \)

1) **Free oscillations**

The perturbation isn’t applied, i.e. \( G(t) = 0 \), which leads to:

\[ \ddot{x} + 2\lambda \cdot \dot{x} + \omega_0^2 \cdot x = 0 \quad (3) \]

As initial condition, we choose to let go of the mass with no initial velocity \( \dot{x}(0) = 0 \) from the position \( x_0 \) with respect to its initial condition. Three distinguishable cases can arise, leading to different solutions (fig. 3):

a) \( \lambda^2 < \omega_0^2 \) **weak damping**:

\[ x = x_0 \cdot e^{-\lambda t} \cdot \cos(\omega t - \varphi) \quad (4) \]

with: \( \omega^2 = \omega_0^2 - \lambda^2 \) and \( \tan \varphi = -\frac{\lambda}{\omega} \)

Define: \( T = \frac{2\pi}{\omega} \) = pseudo-period and \( \Delta = \lambda \cdot T \) = logarithmic decay

In particular: \( \lambda = 0 \) (no friction), this is the standard **harmonic oscillator**.

\[ x = x_0 \cos(\omega_0 t - \varphi_0) \]

\( \omega_0 = \sqrt{\frac{k}{m}} \) is the angular frequency of the oscillator, \( \varphi_0 = \) random phase shift)
b) \( \lambda^2 = \omega_0^2 \) **critical damping**

\[ x(t) = x_0 e^{-\lambda t} (1 + \lambda t) \]  

(5)

c) \( \lambda^2 > \omega_0^2 \) **strong damping**: non periodic motion

\[ x = x_0 e^{-\lambda t} (C_1 e^{\omega t} + C_2 e^{-\omega t}) \]  

(6)

With:

\[ \omega^2 = \lambda^2 - \omega_0^2 \quad C_1 = \frac{\omega - \lambda}{2\omega} \quad C_2 = \frac{\omega + \lambda}{2\omega} \]

2). **Forced oscillations**

Suppose the system is initially at rest \( (x(0) = \dot{x}(0) = 0) \), and you apply a sinusoidal perturbation \( (p = p \sin(\Omega t)) \). To simplify the expression, the phase shift is chosen to be zero (by redefining the time \( t = 0 \))

The solution to equation (2) is:

\[ \theta(t) = A(\Omega) \sin(\Omega t - \psi) + Ce^{-\lambda t} \cos(\omega t - \varphi) \quad \text{if} \quad \lambda^2 < \omega_0^2 \quad \text{weak damping} \]

\[ \theta(t) = A(\Omega) \sin(\Omega t - \psi) + e^{-\lambda t} (C_1 + C_2 t) \quad \text{if} \quad \lambda^2 = \omega_0^2 \quad \text{critical damping} \]

\[ \theta(t) = A(\Omega) \sin(\Omega t - \psi) + e^{-\lambda t} (C_1 e^{\omega t} + C_2 e^{-\omega t}) \quad \text{if} \quad \lambda^2 > \omega_0^2 \quad \text{strong damping} \]

permanent oscillations \hspace{2cm} \text{transitory oscillations}

with

\[ \psi = \arctan \left( \frac{2 \cdot \lambda \cdot \Omega}{\omega^2 - \Omega^2} \right) \quad \text{et} \quad A(\Omega) = \frac{p}{\sqrt{(\omega^2 - \Omega^2)^2 + 4 \cdot \lambda^2 \cdot \Omega^2}} \]  

(7)

Fig. 3: Free oscillations in three different cases: weak, critical and strong damping.
The constants \( C \) and \( \varphi \) or \( C_1 \) and \( C_2 \) are determined by the initial conditions in both cases.

The amplitude of the stationary forced oscillations \( A(\Omega) \) as well as the phase shift \( \psi(\Omega) \) depend on the value of \( \Omega \) (Fig. 4).

**Fig. 4:** Amplitude and phase shift of oscillations in a forced regime.

**a) weak damping:**

A typical resonance curve \( \Omega \) is characterized by:

\[
\Omega_r = \sqrt{\omega_0^2 - 2\lambda^2} \quad \text{et} \quad \Delta\Omega = \Omega_2 - \Omega_1 = \frac{2\lambda \omega}{\Omega_r}
\]

- resonant (angular) frequency
- peak width

\( \Omega_1 \) and \( \Omega_2 \) correspond to the width of the peak at mid height, i.e. the value of the amplitude \( \frac{A_{\text{max}}}{\sqrt{2}} \), also given by an attenuation of 3dB with respect to the maximal value.

The Q-factor of the resonance is given by:

\[
Q = \frac{\Omega_r}{\Delta\Omega} = \frac{\Omega_r^2}{2\lambda \omega}
\]
Comment: Les frequencies $\omega_0$, $\omega$ and $\Omega_t$ are such that: 
$$\omega_0^2 - \omega^2 = \omega^2 - \Omega_t^2 = \lambda^2$$

In the absence of friction ($\lambda = 0$), the resonance curve $A(\Omega)$ is made of a singularity at $\Omega = \omega_0$ only (Fig. 5).

**Fig. 5:** Amplitude of the undamped forced oscillations

**Transition period.**

The second term of the equations in (7) represent *own damped oscillations* of angular frequency $\omega$. As $t$ increases, this term quickly loses importance with respect to the first term, which represents stationary forced oscillations of angular frequency $\Omega$ and phase shift $\psi$. As $\Omega$ and $\omega$ are close, the system oscillates at the *beat frequency* $\omega_B = |\omega - \Omega|$.

**b) strong damping:** 
$$\lambda^2 > \frac{\omega_0^2}{2}$$

The curve is maximal in $\Omega = 0$: there are no forced oscillations, but a fixed displacement equal to $\frac{P}{\omega_0}$. The solutions of the mechanical damped oscillator apply directly to the oscillating RCL circuit’s differential equation (1). These oscillators have the following analogies:

<table>
<thead>
<tr>
<th>&quot;electrical&quot;</th>
<th>&quot;mechanical&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0 = \frac{1}{\sqrt{L \cdot C}}$</td>
<td>$\sqrt{\frac{k}{m}}$</td>
</tr>
<tr>
<td>$2\lambda = \frac{1}{R \cdot C}$</td>
<td>$\eta \frac{m}{m}$</td>
</tr>
<tr>
<td>$U$</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Therefore, by studying an RCL circuit, we can observe and simulate a mechanical oscillator. In particular, for $U_p = 0$, if we give a current impulse to the system at $t = 0$, we will have results for “free oscillations”.

For $\frac{U_p}{L \cdot C} = p \sin \Omega t$, we will be observing results for “forced oscillations”.
III. REQUESTED EXPERIMENT

1) Setup the RLC circuit from figure 1 with:
\[ L = \text{variable inductance up to } 1 \ \text{H} \]
\[ C = \text{variable capacitor up to } 0.22 \ \mu\text{F} \]
\[ R = \text{variable resistance up to } 1 \ \text{M} \Omega \]

A function generator is used to supply the circuit.
An oscilloscope is used to read and measure the signal.
A phasemeter is connected between the source (ref) and the output signal \( U(t) \) (sig) allowing to measure the phase shift.
A voltmeter measures all effective voltages.

2) Demonstrate non forced oscillation

Choose a resonant frequency of approximately 7 kHz (\( L \approx 5 \cdot 10^{-2} \ \text{H}, C \approx 10^{-2} \ \mu\text{F} \)) with \( R \approx 900 \ \text{k} \Omega \)

Turn the function generator on “square wave”, with a low frequency of 100 Hz.

Explain what is observed on the oscilloscope, and verify
Reduce the resistance progressively.
Demonstrate weak, critical and strong damping
Discuss qualitatively
For weak damping, measure the damping coefficient \( \lambda \). Compare this value to the theoretical one. Do this calculation for 2 or 3 different circuits (varying \( R, C \) and \( L \)).

3) Demonstrate forced oscillation

Use the NI ELVIS platform as a function generator for “sine waves”. Connect CH0 to measure \( U_{\text{in}} \) and CH1 to measure \( U_{\text{out}} \) (Fig. 6).

Calculate the resonant frequency of the circuit.
Measure and plot \( A(\Omega) \) and \( \psi(\Omega) \) using the LabVIEW program “Bode Analyzer” on the computer (use a sufficient number of steps per decade). Determine the resonant frequency experimentally (cursors), and compare with the theoretical result.

Explain the differences, and recalculate the characteristic values of the circuit by accounting for an internal resistance \( R_{\text{in}} \) in the coil

\[ \omega_0^2 = \omega_0^2 \left(1 + \frac{R_{\text{in}}}{R}\right) \quad \lambda' = \lambda + \frac{R_{\text{in}}}{2L} \]

(To be shown by reader)

Determine \( Q \) and \( \lambda' \) and compare with the value of \( \lambda \) determined for free oscillations.
Determine \( \Omega_r \) and \( Q \) for different experimental conditions (different values of \( R, C \) and \( L \)).

What happens if you exchange the position of the inductance \( L \) and capacitor \( C \) in the circuit?
Fig. 6: Image of the setup