A4. Free and Forced Oscillations

I. OBJECTIVE OF THE EXPERIMENT

Physics classes are generally divided into chapters: mechanics, Thermodynamics, electrodynamics, etc. One interesting aspect can be noted: the equations appear describing different chapters of physics or science in general are often exactly the same. Therefore a given phenomenon often has analogous phenomena. As an example, the propagation of light in a vacuum or matter and the propagation of the acoustic waves; Ohm's law in electricity, Fick's law in thermodynamics (heat conduction) and Darcy's law in engineering technology (diffusion of water in a porous material). Consequently, a study about the phenomenon and the laws describing it leads to interpretations that can be directly applied to phenomena governed by equations of the same form. We can also use this information to simulate technically challenging conditions on a simpler system. Often, a theory is based on a model, and some approximations, both of which can overlap. A model is a schematic representation of what we believe to be real (billiard ball model for ideal gases, for instance). The established model is used as a basis for calculations, that lead to results close to our experience. During the calculation process, it is sometimes necessary to use approximations, i.e. ignore certain aspects of the problem on hand.

This is the mindset that should be held during the following experiment, that will focus on mechanical oscillations.

II. THE MODEL: OSCILLATING DISK

The system that will be studied in this experiment is a rotating disk of inertial moment \( I \) (Fig. 1). The motion can entirely be described by its rotation angle \( \theta \).

It undergoes viscous friction, and can be excited by a time-dependent external perturbation \( G(t) \).

Fig 1: Diagram of the experimental principle
When applying the dynamic equations of the rigid body (theorem of angular momentum) we get the following equation of motion:

\[ I \ddot{\theta} + N \dot{\theta} + k \theta = G(t) \quad (1) \]

with:
- \( I \) = inertial moment of the disk with respect to its axis \([\text{kg} \cdot \text{m}^2]\)
- \( k \) = spring constant \([\text{kg} \cdot \text{m}^2/\text{s}^2]\)
- \( N \) = Friction coefficient \([\text{kg} \cdot \text{m}^2/\text{s}]\)

The solution to equation (1) is standard. It can be found in any general mechanics or calculus lecture. Here, we will give the main results that satisfy the equation, as well as the corresponding physical interpretations of our rotating pendulum.

If we define:
- \( \omega_0 = \sqrt{\frac{k}{I}} \) angular velocity of the harmonic oscillator \([s^{-1}]\) \quad (2)
- \( G(t) = p \sin(\omega t) \) a sinusoidal perturbation of any frequency \( \Omega \) and amplitude \( p \).

The differential equation becomes:

\[ \ddot{\theta} + 2\lambda \dot{\theta} + \omega_0^2 \theta = p \cdot \sin(\Omega \cdot t) \quad (3) \]

1) FREE OSCILLATIONS

No external perturbation is applied: \( p = 0 \).

As our initial condition, we will choose to let go of the disk \((\dot{\theta}(0) = 0)\) from a position \(\theta(0) = \theta_0\) with respect to its initial position.

This yields three distinguishable cases (see Fig. 2):

a) \( \lambda^2 < \omega_0^2 \) weak damping.

\[ \theta = \theta_0 e^{-\lambda t} \cos(\omega_0 t - \varphi) \quad (4) \]

with \( \omega^2 = \omega_0^2 - \lambda^2 \).

We define: \( T = \frac{2\pi}{\omega} \) pseudo-period and \( \Delta = \lambda \cdot T \) logarithmic decay

Special case: \( \lambda = 0 \) (no friction) \(< - > \) harmonic oscillator

\[ \theta = \theta_0 \cos(\omega_0 t - \varphi) \quad (4\text{bis}) \]

\( \omega_0 = \sqrt{\frac{k}{I}} \) is the angular frequency of the oscillator.
b) \( \lambda^2 = \omega_0^2 \)  critical damping

\[ \theta(t) = \theta_0 e^{-\lambda t} (1 + \lambda t) \]  \hspace{1cm} (5)

c) \( \lambda^2 > \omega_0^2 \)  strong damping, non-periodic motion

\[ \theta = \theta_0 e^{-\lambda t} (C_1 e^{\omega t} + C_2 e^{-\omega t}) \]  \hspace{1cm} (6)

with \( \omega^2 = \lambda^2 - \omega_0^2 \)

\begin{center}
\textbf{Fig 2:} Oscillations for different kind of damping
\end{center}

2) **FORCED OSCILLATIONS**

Supposing the system is initially at rest (\( \theta_0 = \dot{\theta}_0 = 0 \)) we apply a sinusoidal perturbation (\( p = p \sin(\Omega t) \)). In order to simplify the expression, we choose a perturbation with no phase-shift. In this case, the differential equation is:

\[ \ddot{\theta} + 2\lambda \cdot \dot{\theta} + \omega_0^2 \cdot \theta = p \cdot \sin(\Omega \cdot t) \]

The solution is:

\[ \theta(t) = A(\Omega) \sin(\Omega t - \varphi) + C e^{-\lambda t} \cos(\omega t - \varphi) \]  si \( \lambda^2 < \omega_0^2 \)  weak damping

\[ \theta(t) = A(\Omega) \sin(\Omega t - \varphi) + e^{-\lambda t} (C_1 + C_2 t) \]  si \( \lambda^2 = \omega_0^2 \)  critical damping

\[ \theta(t) = A(\Omega) \sin(\Omega t - \varphi) + e^{-\lambda t} (C_1 e^{\omega t} + C_2 e^{-\omega t}) \]  si \( \lambda^2 > \omega_0^2 \)  strong damping

permanent motion \hspace{1.5cm} transitional motion \hspace{1.5cm} (7)
\[ \psi = \arctan\left( \frac{2 \cdot \lambda \cdot \Omega}{\omega^2 - \Omega^2} \right) \quad \text{et} \quad A(\Omega) = \frac{p}{\sqrt{\left(\omega^2 - \Omega^2\right)^2 + 4 \cdot \lambda^2 \cdot \Omega^2}} \] (8)

The constants \((C \text{ and } \varphi)\) or \((C_1 \text{ and } C_2)\) are determined by the initial conditions.

The amplitude of the forced stationary oscillations \(A(\Omega)\) as well of the phase-shift \(\psi(\Omega)\) depend of the value of \(\Omega\) (Fig. 3).

\[ A(\Omega) \]

\[ \psi(\Omega) \]

**Fig 3:** The oscillation amplitude \(A(\Omega)\) and phase-shift \(\psi(\Omega)\).

**a) weak damping:**

The amplitude \(A(\Omega)\) is a typical "resonance curve" characterized by:

\[ \Omega_1 = \sqrt{\omega^2 - 2 \lambda^2} \quad \text{et} \quad \Delta \Omega = \Omega_2 - \Omega_1 = \frac{2 \lambda \omega}{\Omega_r} . \]

(resonant frequency) (bandwidth)

\(\Omega_1\) and \(\Omega_2\) correspond to the width of the peak at mid-height, i.e. for an amplitude of \(A_{\text{max}}\), which is also attenuation of \(3 \text{ [dB]}\) with respect to the maximal value

The Q-factor of the resonance is:

\[ Q = \frac{\Omega_r}{\Delta \Omega} = \frac{\Omega_1^2}{2 \lambda \omega} \]

**Comment:** The frequencies \(\omega_0, \omega \text{ and } \Omega_1\) are such that: \(\omega_0^2 - \omega^2 = \omega^2 - \Omega_1^2 = \lambda^2\)
Without damping \((\lambda = 0)\), the resonance curve \(A(\Omega)\) shows only a singularity for \(\Omega = \omega_0\) (Fig. 4).

**Transitional motion.**

The second term of equation (7) describes damped eigenmodes of frequency \(\omega\). As \(t\) increases, it loses importance with respect to the first term, that describes forced stationary modes of frequency \(\Omega\) and phase-shift \(\psi\). When \(\Omega\) and \(\omega\) are close (Fig. 5), the system oscillates with a beat frequency \(\omega_b = |\omega - \Omega|\). Notice that the beat period is zero for \(\Omega = \omega\).

**b) strong damping:**

\[ \lambda^2 > \frac{\omega_0^2}{2} \]

The curve has a maximum for \(\Omega = 0\): there are no forced oscillations, but a fixed displacement equal to \(\frac{p}{\omega_0}\).

**III. DESCRIPTION OF THE EXPERIMENTAL SETUP**

The setup (Fig. 6) is made of secured system connected to an \(x\)-\(y\)-\(t\) plotter and a frequency or time counter. A photo of the system is presented in Fig. 7.
System (front)

1. Main switch
2a. Magnetic damping (control)
2b. Magnetic damping (system). This allows for the damping to be altered
3. Potentiometer 1: measure of the oscillation amplitude of the disk
   -> connect to the x input of the plotter
4. Potentiometer 2: measure of the perturbation amplitude
   -> connect to the y input of the plotter
5a. Motor allowing to force the oscillation of the disk.
5b. Motor switch 5a
5c. Control for the motor 5a (changes the motor’s speed)

6. Disk
7. Spring + transmission of the perturbation
8a. Phase shift detection between perturbation (motor) and response (disk).
   This system is made of two gates that detect when the perturbation and the disk pass through. Electric pulses are sent to a time counter (electronic stopwatch).
   The HP 5315 A counter allows us to measure:
   - the perturbation frequency or period
   - the frequency or period of the disk (response) and the phase shift (see below).
8b. Switch to choose the output towards the counter HP 5315 A. A diagram of the output signal with respect to the switch position is available below. The correct settings of the counter to measure the different times are detailed in documentation available on location.

IV. SUGGESTED WORK

1) By using the equations of Newtonian mechanics, show that the equation of motion of the oscillating disk is indeed given by equation (1).

2) Determine the damping coefficient \( \lambda \) and the frequency \( \omega \) for free oscillations.
   For this, set the magnetic damping to a maximum. Use the tracker, and choose the \( y, t \) appropriately. Before starting a measurement, make sure the equilibrium position is at 0°. It this is not the case, it can be achieved by delicately moving the motor arm. The angle \( \theta_0 \) for \( t = 0 \) should be close to 135°.
   Can we determine the frequency \( \omega_0 \) of the ideal frictionless oscillator?
   Repeat the experiment without magnetic damping. Can we determine a value of \( \omega \) and \( \lambda \)? Discuss.

3) For weak damping, highlight the beat frequency of the forced oscillator. The oscillating frequency, the perturbation frequency and the beat frequency, respectively \( \omega \), \( \Omega \) and \( \omega_b \) will be determined experimentally, verify the relation \( \omega_b = |\omega - \Omega| \).
   The frequencies \( \omega_0 \), \( \omega \) and \( \Omega \), can be measured using the counter or the plotter. \( \omega_b \) can be measured from the plotter.

4) For a weak electromagnetic damping (in order to reduce the transitional beat phenomena) study the amplitude \( A(\Omega) \) and the phase shift \( \psi(\Omega) \) of the permanent motion.
   Experimentally measure the resonance frequency, the Q-factor and the bandwidth. The inertial moment of the disk can be modified using weights. Be sure to balance the weights well around the disk, to avoid damaging the equipment.

5) Using the obtained results, talk about the different parameters that can modify the resonant frequency of a system.
Fig 7: Photo of the experimental setup